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Quantum energy teleportation with an electromagnetic field: discrete versus continuous variables

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Abstract
It is well known that usual quantum teleportation protocols cannot transport energy. Recently, new protocols called quantum energy teleportation (QET) have been proposed, which transport energy by local operations and classical communication with the ground states of many-body quantum systems. In this paper, we compare two different QET protocols for transporting energy with the electromagnetic field. In the first protocol, a 1/2 spin (a qubit) is coupled with the quantum fluctuation in the vacuum state and measured in order to obtain one-bit information about the fluctuation for the teleportation. In the second protocol, a harmonic oscillator is coupled with the fluctuation and measured in order to obtain continuous-variable information about the fluctuation. In the spin protocol, the amount of teleported energy is suppressed by an exponential damping factor when the amount of input energy increases. This suppression factor becomes power damping in the case of the harmonic oscillator protocol. Therefore, it is concluded that obtaining more information about the quantum fluctuation leads to teleporting more energy. This result suggests a profound relationship between energy and quantum information.

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1. Introduction

In quantum field theory, the concept of negative energy physics has attracted considerable attention for a long time. Quantum interference can produce various states containing regions of negative energy, although the total energy remains nonnegative [1]. The concept of negative energy plays important roles in many fundamental problems of physics, including traversable wormhole [2], cosmic censorship [3] and the second law of thermodynamics [4]. In addition, its physical application to quantum optics has been discussed [5]. Recently, negative energy physics has yielded a quantum protocol called quantum energy teleportation (QET) in which energy can be transported using only local operations and classical communication (LOCC) without breaking causality and local energy conservation [6–8]. QET can be theoretically considered in various many-body quantum systems including (1 + 1)-dimensional massless fields [6], spin chains [7] and cold trapped ions [8]. Based on developing measurement technology with sensitive energy resolution for the systems, the QET effect might be observable in future. It may also be possible to enhance the amount of teleported energy by preparing a large number of parallel QET channels, performing a QET protocol for each channel and accumulating each teleported energy so as to achieve the desired amount of total energy. After future experimental verification of QET, an amazing possibility would be open in principle for nano-technology application of QET. For example, it may be imagined that, without heat generation in the intermediate subsystems of the QET channels, energy is transported in nano-machines at a speed much faster than the evolution speed of excitations of the channels. This technology, if possible, helps the future development of quantum computers in which energy distribution and quantum tasks in the devices are completed before heat generation in the system. QET is also expected to provide insights on unsolved problems in gravitational physics. In fact, a QET process has already been analyzed in black hole physics, and from the measured information of zero-point oscillation of quantum fields, it can be regarded as controlled black hole evaporation if we consider the protocol near the horizon of a large-mass black hole [9].

Energy transportation usually requires physical carriers of energy such as electric currents and radiation waves. Energy is infused into the gateway point of a transport channel connected to a distant exit point. Then, energy carriers of the channel excite and propagate to the exit point. At the exit point, energy is extracted from the carriers and harnessed for many purposes. On the other hand, in the QET protocols, energy can be extracted from the exit point even if no excited energy carriers arrive at the exit point of the channel. We locally measure quantum fluctuation around the gateway point in the ground state of the channel system and announce the measurement result to the distant exit point with zero energy density, where we can extract energy from the channel. A key feature is that this measurement result includes information about the quantum fluctuation of the channel around this distant point via quantum correlation of the ground state of the channel system. Therefore, we can infer details about the behavior of a distant fluctuation from the result of the local measurement. To compensate the extraction of this information, some amount of energy must be infused into the channel system at the measurement point; this is regarded as input energy to the gateway point of the channel. By choosing and performing a proper local operation based on the announced information at the distant point, the local zero-point oscillation around the distant point can be suppressed relative to the ground-state one, yielding a negative energy density. During the operation, respecting local energy conservation, positive amount of surplus energy is moved from the channel system to external systems. This is regarded as output teleported energy from the exit point of the channel.

One of the important unresolved problems in QET is the theoretical clarification of the properties in 1 + 3 dimensions. Protocols in 1 + 1 dimensions have already been extensively analyzed in previous studies [6–8]. However, (1 + 3)-dimensional models have not yet been analyzed. In addition, all the protocols proposed thus far adopt quantum measurements for discrete-variable information. Therefore, it would be interesting to investigate not only a protocol with discrete-variable information but also one with continuous-variable information. In this study, we carry out a detailed analysis of two QET protocols for (1 + 3)-dimensional electromagnetic field in the Coulomb gauge. Local measurements of quantum fluctuations in the vacuum state of the field require energy infusion to the field. The infused energy is diffused to spatial infinity at the velocity of light and the state of the field soon becomes a local vacuum with zero energy around the measurement area. Obviously, this escaped energy cannot be taken back to the measurement area by local operations around this area if we do not know the measurement result of the fluctuation. However, if the measurement result is available, we can effectively take back a part of this energy to the measurement area by applying the QET mechanism. By carrying out a local unitary operation dependent on the measurement result for the measurement area with zero energy density, the fluctuation of zero-point oscillation is squeezed and a negative energy density appears around the area, accompanied by the extraction of positive energy from the fluctuation to external systems. Needless to say, without the measurement result, it is impossible to extract energy from the zero-energy fluctuation. One of the two QET protocols we will consider is a teleportation in which discrete-variable information about a fluctuation is obtained using a measurement with a 1/2 spin (a qubit), and the other is a teleportation in which continuous-variable information is obtained using a measurement with a harmonic oscillator. The discrete-variable protocol is a straightforward extension of the protocol for a $(1 + 1)$-dimensional field proposed in [6]. The measurements are generalized (POVM) ones that use probe systems (1/2 spin and harmonic oscillator) strongly interacting with local electric field fluctuations during a short time. We prove that for a large energy input, the continuous-variable teleportation is more attractive than the discrete-variable teleportation. In the discrete-variable case, the amount of teleported energy is suppressed by an exponential damping factor when the amount of energy infused by the measurement increases. Meanwhile, this suppression factor becomes power damping in the continuous-variable case. Therefore, it is concluded that obtaining more information about the quantum fluctuation leads to teleporting more energy. This result suggests a new profound relation between energy and quantum information. So far, the relationship between energy and information has been extensively discussed only in the context of computation energy cost [10–12]. The QET viewpoint may shed light on a new relationship between the amount of teleported energy and the amount of quantum information about ground-state fluctuations which would be characterized by various informational indices including mutual information and entanglement. The explicit analysis about this relationship is beyond the scope of this paper and will be reported elsewhere.

The remainder of this paper is organized as follows. In section 2, a brief review of the quantization of the electromagnetic field in the Coulomb gauge is presented in order to clarify our notations, and the emergence of negative energy density is explained. In section 3, we discuss a discrete-variable protocol. In section 4, a continuous-variable protocol is analyzed. In section 5, a summary and discussions are presented. In this paper, we adopt the natural unit $c = \hbar = 1$.

2. Quantization in Coulomb gauge

We present a short review of quantization of the electromagnetic field in the Coulomb gauge in order to clarify the notations used for later discussions. The gauge is defined by

$$A_0 = 0, \qquad \mathrm{div}\,\mathbf{A} = 0$$

for the gauge field $A_\mu = (A_0, \mathbf{A})$. Then, the equation of motion of the Heisenberg operator of the gauge field $\hat{\mathbf{A}}(t, \mathbf{x})$ is reduced to the massless Klein–Gordon equation given by

$$\left[\partial_t^2 - \nabla^2\right]\hat{\mathbf{A}} = 0.$$

The solution can be expanded in terms of plain-wave modes as follows:

$$\hat{\mathbf{A}}(t, \mathbf{x}) = \int \frac{\mathrm{d}^3 k}{\sqrt{(2\pi)^3 2|\mathbf{k}|}} \sum_{h=1,2} \left[\mathbf{e}_h(\mathbf{k})\hat{a}_{\mathbf{k}}^h \mathrm{e}^{\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|t)} + \mathbf{e}_h^*(\mathbf{k})\hat{a}_{\mathbf{k}}^{h\dagger} \mathrm{e}^{-\mathrm{i}(\mathbf{k}\cdot\mathbf{x}-|\mathbf{k}|t)}\right],$$

where $\hat{a}_{\mathbf{k}}^{h\dagger}\left(\hat{a}_{\mathbf{k}}^{h}\right)$ is a creation (annihilation) operator of the photon with momentum $\mathbf{k}$ and polarization h satisfying

$$\left[\hat{a}_{\mathbf{k}}^{h}, \hat{a}_{\mathbf{k}'}^{h'\dagger}\right] = \delta_{hh'}\delta(\mathbf{k}-\mathbf{k}'),$$

and $\mathbf{e}_h(\mathbf{k})$ is a polarization vector satisfying

$$\mathbf{e}_h(\mathbf{k})^* \cdot \mathbf{e}_{h'}(\mathbf{k}) = \delta_{hh'},$$
$$\mathbf{k} \cdot \mathbf{e}_h(\mathbf{k}) = 0.$$

In this study, because we take a sum of two polarization contributions to obtain the final results, the reality condition can be imposed on $\mathbf{e}_h(\mathbf{k})$ for simplicity such that $\mathbf{e}_h^*(\mathbf{k}) = \mathbf{e}_h(\mathbf{k})$. In addition, $\mathbf{e}_h(\mathbf{k})$ satisfies the completeness relation as

$$\sum_h e_h^a(\mathbf{k}) e_h^b(\mathbf{k}) = \delta_{ab} - \frac{k_a k_b}{\mathbf{k}^2}.$$

The energy density operator of the field is defined by

$$\hat{\varepsilon}(\mathbf{x}) = \frac{1}{2} : (\hat{\mathbf{E}}(\mathbf{x})^2 + (\nabla \times \hat{\mathbf{A}}(\mathbf{x}))^2) :,$$

where $\hat{\mathbf{E}}(\mathbf{x})$ is the electric field operator and :: denotes the standard normal order with respect to $\hat{a}_{\mathbf{k}}^{h}$ and $\hat{a}_{\mathbf{k}}^{h\dagger}$. The Hamiltonian is given by the spatial integration of the energy density as follows:

$$\hat{H} = \int \hat{\varepsilon}(\mathbf{x})\, \mathrm{d}^3 x.$$

It is a well-known fact that the Hamiltonian (total energy of the field) is a nonnegative operator. The vacuum state $|0\rangle$ is the eigenstate with the lowest eigenvalue zero of $\hat{H}$ as $\hat{H}|0\rangle = 0$. The expectation value of energy density vanishes for the vacuum state as

$$\langle 0|\hat{\varepsilon}(\mathbf{x})|0\rangle = 0. \tag{1}$$

In the later discussion, coherent states are used often. Therefore, we present a summary of the related properties of the coherent states. A displacement operator generating a coherent state from $|0\rangle$ is given by

$$\hat{U}(\mathbf{p}, \mathbf{q}) = \exp\left[\mathrm{i}\int [\mathbf{p}(\mathbf{x}) \cdot \hat{\mathbf{A}}(\mathbf{x}) - \mathbf{q}(\mathbf{x}) \cdot \hat{\mathbf{E}}(\mathbf{x})]\mathrm{d}^3 x\right],$$

where $\mathbf{p}(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ are real vector functions satisfying the conditions of this gauge as

$$\nabla \cdot \mathbf{p}(\mathbf{x}) = 0,$$
$$\nabla \cdot \mathbf{q}(\mathbf{x}) = 0.$$

By using the commutation relation between the gauge field and the electric field at time $t = 0$ given by

$$[\hat{A}_a(\mathbf{x}), \hat{E}_b(\mathbf{y})] = \mathrm{i}\left(\delta_{ab} - \frac{\partial_a \partial_b}{\nabla^2}\right)\delta(\mathbf{x}-\mathbf{y}),$$

it is easily verified that $\hat{U}(\mathbf{p}, \mathbf{q})$ displaces $\hat{\mathbf{A}}(\mathbf{x})$ and $\hat{\mathbf{E}}(\mathbf{x})$ as

$$\hat{U}^\dagger(\mathbf{p}, \mathbf{q})\hat{\mathbf{E}}(\mathbf{x})\hat{U}(\mathbf{p}, \mathbf{q}) = \hat{\mathbf{E}}(\mathbf{x}) + \mathbf{p}(\mathbf{x}), \tag{2}$$

$$\hat{U}^\dagger(\mathbf{p}, \mathbf{q})\hat{\mathbf{A}}(\mathbf{x})\hat{U}(\mathbf{p}, \mathbf{q}) = \hat{\mathbf{A}}(\mathbf{x}) + \mathbf{q}(\mathbf{x}). \tag{3}$$

In addition, we are able to prove a product formula as

$$\hat{U}(\mathbf{p}_1, \mathbf{q}_1)\hat{U}(\mathbf{p}_2, \mathbf{q}_2) = \exp\left[\frac{\mathrm{i}}{2}\int (\mathbf{p}_1 \cdot \mathbf{q}_2 - \mathbf{q}_1 \cdot \mathbf{p}_2)\mathrm{d}^3 x\right]\hat{U}(\mathbf{p}_1 + \mathbf{p}_2, \mathbf{q}_1 + \mathbf{q}_2).$$

This implies that the set of $\hat{U}(\mathbf{p}, \mathbf{q})$ forms a unitary ray representation of the displacement group of the field. Coherent states generated by $\hat{U}(\mathbf{p}, \mathbf{q})$ are defined by

$$|(\mathbf{p}, \mathbf{q})\rangle = \hat{U}(\mathbf{p}, \mathbf{q})|0\rangle. \tag{4}$$

By using the Fourier transformation of $\mathbf{p}(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ defined by

$$\mathbf{P}(\mathbf{k}) = \int \mathbf{p}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}^3x,$$
$$\mathbf{Q}(\mathbf{k}) = \int \mathbf{q}(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}^3x,$$

the coherent states are explicitly written in terms of the creation operator $\hat{a}_{\mathbf{k}}^{h\dagger}$ as follows:

$$|(\mathbf{p}, \mathbf{q})\rangle = \exp\left[-\frac{1}{2}\int \frac{\mathrm{d}^3k}{(2\pi)^3 2|\mathbf{k}|}|\mathbf{P}(\mathbf{k}) - \mathrm{i}|\mathbf{k}|\mathbf{Q}(\mathbf{k})|^2\right]$$
$$\times \exp\left[\mathrm{i}\int \frac{\mathrm{d}^3k}{\sqrt{(2\pi)^3 2|\mathbf{k}|}}\sum_h \mathbf{e}_h(\mathbf{k})\cdot(\mathbf{P}(\mathbf{k}) - \mathrm{i}|\mathbf{k}|\mathbf{Q}(\mathbf{k}))a_{\mathbf{k}}^{h\dagger}\right]|0\rangle.$$

From the above expression, it is easy to prove that $|(\mathbf{p}, \mathbf{q})\rangle$ is an eigenstate of the annihilation operator $\hat{a}_{\mathbf{k}}^{h}$ such that

$$\hat{a}_{\mathbf{k}}^{h}|(\mathbf{p}, \mathbf{q})\rangle = \frac{\mathrm{i}}{\sqrt{(2\pi)^3 2|\mathbf{k}|}}\mathbf{e}_h(\mathbf{k})\cdot(\mathbf{P}(\mathbf{k}) - \mathrm{i}|\mathbf{k}|\mathbf{Q}(\mathbf{k}))|(\mathbf{p}, \mathbf{q})\rangle. \tag{5}$$

The inner product of two coherent states is explicitly calculated as

$$\langle(\mathbf{p}_1, \mathbf{q}_1)|(\mathbf{p}_2, \mathbf{q}_2)\rangle = \mathrm{e}^{\frac{\mathrm{i}}{2}\int(\mathbf{p}_1\cdot\mathbf{q}_2 - \mathbf{q}_1\cdot\mathbf{p}_2)d^3x}$$
$$\times\, \mathrm{e}^{-\frac{1}{2}\int\frac{d^3k}{(2\pi)^3 2|\mathbf{k}|}|\mathbf{P}_1(\mathbf{k}) - \mathbf{P}_2(\mathbf{k}) - \mathrm{i}|\mathbf{k}|(\mathbf{Q}_1(\mathbf{k}) - \mathbf{Q}_2(\mathbf{k}))|^2}. \tag{6}$$

Next, we examine the emergence of a region with negative energy density in this standard theory. As a simple example [4], let us consider a superposition state $|\Psi\rangle$ of the vacuum state $|0\rangle$ and a two-photon state $|2\rangle$ such that

$$|\Psi\rangle = \cos\theta|0\rangle + \mathrm{e}^{\mathrm{i}\delta}\sin\theta|2\rangle,$$

where θ and δ are real parameters with $\theta \in [0, \pi]$ and $\delta \in [0, 2\pi]$. Generally, an off-diagonal element of the energy density $\langle 0|\hat{\varepsilon}(\mathbf{x})|2\rangle$ does not vanish for a fixed point $\mathbf{x}$. This is because $\hat{\varepsilon}(\mathbf{x})$ includes a non-vanishing term proportional to $\hat{a}_{\mathbf{k}}^{h}\hat{a}_{\mathbf{k}'}^{h'}$. This fact indicates the emergence of negativity of the energy density as follows. The expectation value of the energy density for the state $|\Psi\rangle$ is calculated as

$$\langle\Psi|\hat{\varepsilon}(\mathbf{x})|\Psi\rangle = 2\cos\theta\sin\theta(\cos\delta\mathrm{Re}\langle 0|\hat{\varepsilon}(\mathbf{x})|2\rangle - \sin\delta\mathrm{Im}\langle 0|\hat{\varepsilon}(\mathbf{x})|2\rangle) + \sin^2\theta\langle 2|\hat{\varepsilon}(\mathbf{x})|2\rangle.$$

In this result, let us set the parameters θ and δ so as to satisfy

$$\cos\theta = \frac{\langle 2|\hat{\varepsilon}(\mathbf{x})|2\rangle}{\sqrt{\langle 2|\hat{\varepsilon}(\mathbf{x})|2\rangle^2 + 4|\langle 0|\hat{\varepsilon}(\mathbf{x})|2\rangle|^2}},$$
$$\sin\theta = \frac{2|\langle 0|\hat{\varepsilon}(\mathbf{x})|2\rangle|}{\sqrt{\langle 2|\hat{\varepsilon}(\mathbf{x})|2\rangle^2 + 4|\langle 0|\hat{\varepsilon}(\mathbf{x})|2\rangle|^2}},$$
$$\tan\delta = -\frac{\mathrm{Im}\langle 0|\hat{\varepsilon}(\mathbf{x})|2\rangle}{\mathrm{Re}\langle 0|\hat{\varepsilon}(\mathbf{x})|2\rangle}.$$

Then, $\langle\Psi|\hat{\varepsilon}(\mathbf{x})|\Psi\rangle$ is evaluated as a negative value as follows:

$$\langle\Psi|\hat{\varepsilon}(\mathbf{x})|\Psi\rangle = -\tfrac{1}{2}\left[\sqrt{\langle 2|\hat{\varepsilon}(\mathbf{x})|2\rangle^2 + 4|\langle 0|\hat{\varepsilon}(\mathbf{x})|2\rangle|^2} - \langle 2|\hat{\varepsilon}(\mathbf{x})|2\rangle\right] < 0.$$

Therefore, the emergence of negative-energy regions is not a peculiar phenomenon in quantum field theory. Quantum interference in the superposition of photon number eigenstates yields negative values. It should be re-emphasized that despite the existence of regions with negative energy density, the expectation values of $\hat{H}$ remain nonnegative. This implies that there exist regions with a sufficient amount of positive energy so as to make the total energy nonnegative. As described in sections 3 and 4, this negative energy plays a crucial role in the QET protocols.

3. Discrete-variable teleportation

Our protocol for QET with a 1/2 spin probe is a straightforward extension of that in [6] and comprises the following three steps.

(1) At time $t = 0$, the spin probe is strongly coupled with the vacuum fluctuation of the electric field within a finite compact region V_m during a very short time. In this process, some information about the fluctuation is imprinted into the probe. Positive energy is inevitably infused into the field during the measurement process, as seen later. The amount of energy is denoted by E_m. When this energy is infused, positive-energy wave packets of the field are generated and these propagate to spatial infinity with the velocity of light.

(2) After switching off the interaction, projective measurement of the z-component of the spin is carried out. If the up or down state is observed, we assign $s = +$ or $-$, respectively, to the measurement result. This implies that we obtain one-bit information about the field fluctuation via the probe measurement.

(3) At time $t = T$, it is assumed that the measurement has finished and the wave packets have already escaped from the region. Hence, the energy density in the region V_m is exactly zero. Then, a local displacement operation is carried out depending on s within V_m. Even though we have zero energy in V_m, positive energy is extracted from the field fluctuation during the local operation, generating negative-energy wave packets of the field. The amount of negative energy of the wave packets is denoted by $E_o(= -|E_o|)$. Therefore, the amount of energy extracted from the field is given by $+|E_o|$.

In step (1), the measurement interaction between the electric field and the spin probe is given by

$$\hat{H}_m(t) = g(t)\hat{\sigma}_z\hat{G}, \tag{7}$$

where $\hat{\sigma}_z$ is the z-component of the Pauli matrices of the spin probe, $g(t)$ is a time-dependent real coupling constant and $\hat{G}$ is a Hermitian operator defined by

$$\hat{G} = \frac{\pi}{4} - \int \mathbf{a}_m(\mathbf{x}) \cdot \hat{\mathbf{E}}(\mathbf{x}) \mathrm{d}^3x. \tag{8}$$

Here, $\mathbf{a}_m(\mathbf{x})$ is a real vector function with a support equal to V_m satisfying $\nabla \cdot \mathbf{a}_m(\mathbf{x}) = 0$. In addition, by taking a short-time limit for switching the interaction, we set

$$g(t) = \delta(t - 0). \tag{9}$$

The initial state of the spin probe is assumed to be the up state $|+_x\rangle$ of the x-component of the spin given by

$$|+_x\rangle = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In step (2), the measurement operators $\hat{M}_\pm$ [13] are defined by

$$\hat{M}_\pm = \langle \pm_x | \mathrm{T} \exp\left[-\mathrm{i}\int_0^{t_m} \hat{H}_m(t)\mathrm{d}t\right] |+_x\rangle, \tag{10}$$

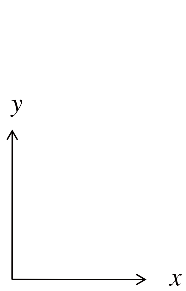

Figure 1. The first schematic diagram of QET in the xy-plane slice. The local measurement is performed with the infusion of positive energy E_m to the field and the measurement result s is obtained. The measurement area V_m is represented by a circle in the plane. A positive-energy wave packet is generated in the field system and it escapes to spatial infinity at the velocity of light.

where $|-_x\rangle$ is the down state of the x-component of the spin given by

$$|-_x\rangle = \frac{-\mathrm{i}}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

By using equation (9), $\hat{M}_\pm$ are computed as

$$\hat{M}_+ = \cos\hat{G}, \tag{11}$$

$$\hat{M}_- = \sin\hat{G}. \tag{12}$$

Using equation (4), the post-measurement states of the field obtaining $s = \pm$ are calculated as

$$|\psi_+\rangle = \frac{1}{\sqrt{p_+}}\hat{M}_+|0\rangle = \frac{1}{2\sqrt{p_+}}[\mathrm{e}^{\mathrm{i}\frac{\pi}{4}}|(\mathbf{0}, \mathbf{a}_m)\rangle + \mathrm{e}^{-\mathrm{i}\frac{\pi}{4}}|(\mathbf{0}, -\mathbf{a}_m)\rangle],$$

$$|\psi_-\rangle = \frac{1}{\sqrt{p_-}}\hat{M}_-|0\rangle = \frac{1}{2\mathrm{i}\sqrt{p_-}}[\mathrm{e}^{i\frac{\pi}{4}}|(\mathbf{0}, \mathbf{a}_m)\rangle - \mathrm{e}^{-i\frac{\pi}{4}}|(\mathbf{0}, -\mathbf{a}_m)\rangle],$$

where $p_\pm$ is the measurement probability of $|\pm_x\rangle$ for the spin probe and it is evaluated as

$$p_\pm = \langle 0|\hat{M}_\pm^\dagger \hat{M}_\pm|0\rangle = \tfrac{1}{2}.$$

After the measurement, the average state of the field is given by

$$\hat{\rho}_{\hat{M}} = \sum_{s=\pm} p_s|\psi_s\rangle\langle\psi_s|.$$

The amount of energy E_m of the field after the measurement is not zero but is instead a positive value given by

$$E_m = \int \mathrm{Tr}[\hat{\rho}_{\hat{M}}\hat{\varepsilon}(\mathbf{x})]\mathrm{d}^3x = \frac{1}{2}\int (\nabla \times \mathbf{a}_m(\mathbf{x}))^2\mathrm{d}^3x. \tag{13}$$

This evaluation of equation (13) is achieved by using equation (5). Because the initial state of the field is the vacuum state and it has zero energy, the positive value in equation (13) implies that the manipulation of the measurement requires positive work from outside. In general, any local operation on the vacuum state infuses nonzero energy into the field if the post-operation state is not the vacuum state, because the Hamiltonian is a nonnegative operator. In figure 1, the measurement process of QET (step (1)) is depicted in the xy-plane slice. The measurement area V_m is represented by a circle in the plane.

Next, let us discuss time evolution of the field after the measurement. The average state evolves as

$$\hat{\rho}_m(t) = \sum_{s=\pm} p_s \hat{U}(t)|\psi_s\rangle\langle\psi_s|\hat{U}(t)^\dagger$$

by the time-evolution operator $\hat{U}(t) = \exp[-it\hat{H}]$. The Heisenberg operators $\hat{\mathbf{A}}(t,\mathbf{x})$ and $\hat{\mathbf{E}}(t,\mathbf{x})$ are given by the Schrödinger operators $\hat{\mathbf{A}}(\mathbf{x})(=\hat{\mathbf{A}}(0,\mathbf{x}))$ and $\hat{\mathbf{E}}(\mathbf{x})(=\hat{\mathbf{E}}(0,\mathbf{x}))$ as follows:

$$\hat{\mathbf{A}}(t,\mathbf{x}) = \int \Delta^{(2)}(t,\mathbf{x}-\mathbf{y})\hat{\mathbf{A}}(0,\mathbf{y})\mathrm{d}^3y + \int \Delta^{(1)}(t,\mathbf{x}-\mathbf{y})\hat{\mathbf{E}}(0,\mathbf{y})\mathrm{d}^3y, \tag{14}$$

$$\hat{\mathbf{E}}(t,\mathbf{x}) = \int \partial_t\Delta^{(2)}(t,\mathbf{x}-\mathbf{y})\hat{\mathbf{A}}(0,\mathbf{y})\mathrm{d}^3y + \int \Delta^{(2)}(t,\mathbf{x}-\mathbf{y})\hat{\mathbf{E}}(0,\mathbf{y})\mathrm{d}^3y, \tag{15}$$

where $\Delta^{(1)}$ and $\Delta^{(2)}$ are Lorentz-invariant functions defined by

$$\Delta^{(1)}(t,\mathbf{x}-\mathbf{y}) = \frac{1}{2\pi}\epsilon(t)\delta(t^2-|\mathbf{x}-\mathbf{y}|^2),$$
$$\Delta^{(2)}(t,\mathbf{x}-\mathbf{y}) = \partial_t\Delta^{(1)}(t,\mathbf{x}-\mathbf{y}). \tag{16}$$

Both the functions $\Delta^{(1)}$ and $\Delta^{(2)}$ vanish in the off-light-cone region with $t^2-|\mathbf{x}-\mathbf{y}|^2 \neq 0$ and satisfy the massless Klein–Gordon equation:

$$\left[\partial_t^2-\nabla^2\right]\Delta^{(n)}(t,\mathbf{x}-\mathbf{y}) = 0.$$

Substituting equation (14) and equation (15) into $\hat{\varepsilon}(t,\mathbf{x})$ yields the time evolution of the average value of the energy density as

$$\langle\hat{\varepsilon}(t,\mathbf{x})\rangle = \mathrm{Tr}[\hat{\rho}_m(t)\hat{\varepsilon}(\mathbf{x})] = \tfrac{1}{2}[\mathbf{\Pi}(t,\mathbf{x})^2 + \mathbf{b}(t,\mathbf{x})^2], \tag{17}$$

where $\mathbf{\Pi}(t,\mathbf{x})$ and $\mathbf{b}(t,\mathbf{x})$ are given by

$$\mathbf{b}(t,\mathbf{x}) = \int \Delta^{(2)}(t,\mathbf{x}-\mathbf{y})\nabla\times\mathbf{a}_m(0,\mathbf{y})\mathrm{d}^3y, \tag{18}$$

$$\mathbf{\Pi}(t,\mathbf{x}) = \int \partial_t\Delta^{(2)}(t,\mathbf{x}-\mathbf{y})\mathbf{a}_m(0,\mathbf{y})\mathrm{d}^3y. \tag{19}$$

Taking account of the explicit form of $\Delta^{(2)}(t,\mathbf{x}-\mathbf{y})$ in equation (16), the above result teaches us that the wave packets excited by the measurement soon escape from the measurement area to spatial infinity with the velocity of light. This ensures that the energy density in V_m returns to zero. At time $t = T$, the state $\hat{\rho}_m(T)$ is assumed to be a local vacuum state with zero energy density:

$$\mathrm{Tr}[\hat{\rho}_m(T)\hat{\varepsilon}(\mathbf{x})] = 0$$

for $\mathbf{x}\in V_m$.

In step (3), let us consider a local displacement operation within V_m depending on s defined by

$$\hat{U}_s = \exp\left[is\theta\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{A}}(\mathbf{x})\mathrm{d}^3x\right], \tag{20}$$

where θ is a real constant fixed below, and $\mathbf{f}_o(\mathbf{x})$ is a real vector function with the support V_m that satisfies

$$\nabla\cdot\mathbf{f}_o(\mathbf{x}) = 0$$

so as to preserve gauge invariance of $\hat{U}_s$. In this process, positive energy is released on average from the field to the apparatus executing $\hat{U}_s$ by taking a proper value of θ. Let us introduce the localized energy operator around V_m as

$$\hat{H}_o = \int \mathbf{w}(\mathbf{x})\hat{\varepsilon}(\mathbf{x})\mathrm{d}^3x,$$

where $\mathbf{w}(\mathbf{x})$ is a real window function of V_m that satisfies $\mathbf{w}(\mathbf{x}) = 1$ for $\mathbf{x}\in V_m$ and decays rapidly outside V_m. The average state after the displacement operation is given by

$$\hat{\rho} = \sum_{s=\pm} \hat{U}_s\hat{U}(T)\hat{M}_s|0\rangle\langle 0|\hat{M}_s^\dagger\hat{U}^\dagger(T)\hat{U}_s^\dagger.$$

For this state, the average energy of the field around V_m is defined by

$$E_o = \mathrm{Tr}[\hat{H}_o\hat{\rho}] = \sum_{s=\pm}\langle 0|\hat{M}_s^\dagger\hat{U}^\dagger(T)\hat{U}_s^\dagger\hat{H}_o\hat{U}_s\hat{U}(T)\hat{M}_s|0\rangle. \tag{21}$$

Here, the operator $\hat{U}^\dagger(T)\hat{U}_s^\dagger\hat{H}_o\hat{U}_s\hat{U}(T)$ is rewritten using equations (2) and (3) as

$$\hat{U}^\dagger(T)\hat{U}_s^\dagger\hat{H}_o\hat{U}_s\hat{U}(T) = \int \mathbf{w}(\mathbf{x})\hat{\varepsilon}(t,\mathbf{x})\mathrm{d}^3x + s\theta\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{E}}(T,\mathbf{x})\mathrm{d}^3x + \frac{1}{2}\theta^2\int \mathbf{f}_o(\mathbf{x})^2\mathrm{d}^3x,$$

where $\hat{\varepsilon}(t,\mathbf{x})$ is the Heisenberg operator given by $\hat{U}^\dagger(T)\hat{\varepsilon}(\mathbf{x})\hat{U}(T)$, and we have used $\mathbf{w}(\mathbf{x})\mathbf{f}_o(\mathbf{x}) = \mathbf{f}_o(\mathbf{x})$. Substituting the above relation into equation (21) yields the following relation:

$$\begin{aligned} E_o &= \sum_{s=\pm}\langle 0|\hat{M}_s^\dagger\left[\int \mathbf{w}(\mathbf{x})\hat{\varepsilon}(t,\mathbf{x})\mathrm{d}^3x\right]\hat{M}_s|0\rangle \\ &\quad + \theta\sum_{s=\pm}s\langle 0|\hat{M}_s^\dagger\left[\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{E}}(T,\mathbf{x})\mathrm{d}^3x\right]\hat{M}_s|0\rangle \\ &\quad + \frac{1}{2}\theta^2\int \mathbf{f}_o(\mathbf{x})^2\mathrm{d}^3x\langle 0|\left(\sum_{s=\pm}\hat{M}_s^\dagger\hat{M}_s\right)|0\rangle. \end{aligned} \tag{22}$$

In order to further simplify the form of E_o, the following equation is used:

$$\begin{aligned} &\left[\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{E}}(T,\mathbf{x})\mathrm{d}^3x, \int \mathbf{a}_m(\mathbf{y})\cdot\hat{\mathbf{E}}(0,\mathbf{y})\mathrm{d}^3y\right] \\ &\quad = \mathrm{i}\int\int \mathbf{f}_o(\mathbf{x})\cdot\mathbf{a}_m(\mathbf{y})\partial_T\Delta^{(2)}(T,\mathbf{x}-\mathbf{y})\mathrm{d}^3x\mathrm{d}^3y. \end{aligned} \tag{23}$$

This is proven by equation (15). Taking account of the supports of $\mathbf{f}_o(\mathbf{x})$ and $\mathbf{a}_m(\mathbf{y})$, it is verified that the integration of the right-hand side in equation (23) is zero because $\partial_T\Delta^{(2)}(T,\mathbf{x}-\mathbf{y})$ vanishes when $T^2 - |\mathbf{x}-\mathbf{y}|^2 \neq 0$. Thus, we obtain the relation

$$\left[\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{E}}(T,\mathbf{x})\mathrm{d}^3x, \int \mathbf{a}_m(\mathbf{y})\cdot\hat{\mathbf{E}}(0,\mathbf{y})\mathrm{d}^3y\right] = 0. \tag{24}$$

In a similar manner, we can show the following relation using equations (14) and (15):

$$\left[\int \mathbf{w}(\mathbf{x})\hat{\varepsilon}(T,\mathbf{x})\mathrm{d}^3x, \int \mathbf{a}_m(\mathbf{y})\cdot\hat{\mathbf{E}}(0,\mathbf{y})\mathrm{d}^3y\right] = 0. \tag{25}$$

From these relations, we can show the commutation relations given by

$$\left[\int \mathbf{w}(\mathbf{x})\hat{\varepsilon}(T,\mathbf{x})\mathrm{d}^3x,\ \hat{M}_s^\dagger\right] = 0,$$

$$\left[\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{E}}(T,\mathbf{x})\mathrm{d}^3x,\ \hat{M}_s^\dagger\right] = 0$$

by recalling that $\hat{M}_s^\dagger$ is a function of $\int \mathbf{a}_m(\mathbf{y})\cdot\hat{\mathbf{E}}(0,\mathbf{y})\mathrm{d}^3y$, as seen in equations (8), (11) and (12). Therefore, we can move the positions of $\hat{M}_s^\dagger$ in the first and second terms of the right-hand side of equation (22) to the right, and rewrite E_o as

$$\begin{aligned} E_o = {}& \langle 0|\int \mathbf{w}(\mathbf{x})\hat{\varepsilon}(t,\mathbf{x})\mathrm{d}^3x\left(\sum_{s=\pm}\hat{M}_s^\dagger\hat{M}_s\right)|0\rangle \\ &+\theta\langle 0|\left(\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{E}}(T,\mathbf{x})\mathrm{d}^3x\right)\left(\sum_{s=\pm}s\hat{M}_s^\dagger\hat{M}_s\right)|0\rangle \\ &+\frac{1}{2}\theta^2\int \mathbf{f}_o(\mathbf{x})^2\mathrm{d}^3x\langle 0|\left(\sum_{s=\pm}\hat{M}_s^\dagger\hat{M}_s\right)|0\rangle. \end{aligned} \tag{26}$$

By using two relations of the measurement operators given by

$$\sum_{s=\pm}\hat{M}_s^\dagger\hat{M}_s = \cos^2\hat{G} + \sin^2\hat{G} = 1,$$

$$\sum_{s=\pm}s\hat{M}_s^\dagger\hat{M}_s = \cos^2\hat{G} - \sin^2\hat{G} = \cos(2\hat{G}),$$

the following expression of E_o is obtained:

$$E_o = \langle 0|\int \mathbf{w}(\mathbf{x})\hat{\varepsilon}(t,\mathbf{x})\mathrm{d}^3x|0\rangle + \theta\eta + \frac{1}{2}\theta^2\xi, \tag{27}$$

where η and ξ are real constants defined by

$$\eta = \langle 0|\left(\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{E}}(T,\mathbf{x})\mathrm{d}^3x\right)\cos(2\hat{G})|0\rangle, \tag{28}$$

$$\xi = \int \mathbf{f}_o(\mathbf{x})^2\mathrm{d}^3x. \tag{29}$$

The first term of the right-hand side of equation (27) vanishes because we can transform it into

$$\begin{aligned} &\langle 0|\int \mathbf{w}(\mathbf{x})\hat{\varepsilon}(T,\mathbf{x})\mathrm{d}^3x|0\rangle \\ &\quad = \langle 0|\hat{U}^\dagger(T)\int \mathbf{w}(\mathbf{x})\hat{\varepsilon}(0,\mathbf{x})\mathrm{d}^3xU(T)|0\rangle \\ &\quad = \int \mathbf{w}(\mathbf{x})\langle 0|\hat{\varepsilon}(\mathbf{x})|0\rangle\mathrm{d}^3x = 0. \end{aligned}$$

Here, we have used $\hat{U}(T)|0\rangle = |0\rangle$ and equation (1). The expression of η in equation (28) can be further simplified by noting that

$$\cos(2\hat{G})|0\rangle = \frac{\mathrm{i}}{2}\left[|(\mathbf{0},2\mathbf{a}_m)\rangle - |(\mathbf{0},-2\mathbf{a}_m)\rangle\right]. \tag{30}$$

By using equation (5) and equation (30), the following relation is derived:

$$\langle 0|\hat{\mathbf{E}}(T,\mathbf{x})\cos(2\hat{G})|0\rangle = -\langle 0|(\mathbf{0},2\mathbf{a}_m)\rangle\int \partial_T^2\Delta(T,\mathbf{x}-\mathbf{y})\mathbf{a}_m(\mathbf{y})\mathrm{d}^3y, \tag{31}$$

where $\Delta(t, \mathbf{x})$ is a Lorentz invariant function defined by

$$\Delta(t, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3|\mathbf{k}|} \cos\left(\mathbf{k}\cdot\mathbf{x} - |\mathbf{k}|t\right).$$

In contrast to $\Delta^{(1)}(t, \mathbf{x})$ and $\Delta^{(2)}(t, \mathbf{x})$, the function $\Delta(t, \mathbf{x})$ does not vanish in the off-light-cone region:

$$\Delta(t, \mathbf{x})|_{t^2-\mathbf{x}^2\neq 0} = -\frac{1}{2\pi^2}\frac{1}{t^2-\mathbf{x}^2}.$$

The factor $\langle 0|(\mathbf{0}, 2\mathbf{a}_m)\rangle$ is real and calculated from equation (6) as

$$\langle 0|\,(\mathbf{0}, \mathbf{2a}_m)\rangle = \exp\left[-\int \frac{d^3k|\mathbf{k}|}{(2\pi)^3}|\tilde{\mathbf{a}}_m(\mathbf{k})|^2\right],$$

where $\tilde{\mathbf{a}}_m(\mathbf{k}) = \int \mathbf{a}_m(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}}d^3x$. Substituting equation (31) into equation (28) gives the final expression of η as

$$\eta = \langle 0|(\mathbf{0}, 2\mathbf{a}_m)\rangle \int\int \partial_T^2\Delta(T, \mathbf{x}-\mathbf{y})\mathbf{f}_o(\mathbf{x})\cdot\mathbf{a}_m(\mathbf{y})d^3x d^3y. \quad (32)$$

Minimization of E_o with respect to θ is achieved by taking the parameter θ as

$$\theta = -\frac{\eta}{\xi}. \quad (33)$$

Substituting the value of θ into equation (27) yields the following expression of E_o:

$$E_o = -\frac{\eta^2}{2\xi}. \quad (34)$$

Substituting equations (29) and (32) into equation (34), we obtain our main result in this section:

$$E_o = -D_q \frac{\left[\int\int \partial_T^2\Delta(T, \mathbf{x}-\mathbf{y})\mathbf{f}_o(\mathbf{x})\cdot\mathbf{a}_m(\mathbf{y})d^3x d^3y\right]^2}{2\int \mathbf{f}_o(\mathbf{x})^2 d^3x}, \quad (35)$$

where D_q is an exponential damping factor with respect to $\mathbf{a}_m$ such that

$$D_q = |\langle 0|(\mathbf{0}, 2\mathbf{a}_m)\rangle|^2 = \exp\left[-2\int \frac{d^3k|\mathbf{k}|}{(2\pi)^3}|\tilde{\mathbf{a}}_m(\mathbf{k})|^2\right]. \quad (36)$$

Because $\xi > 0$, as seen in equation (29), it is of importance that E_o is negative in equation (34):

$$E_o = -|E_o| < 0.$$

Respecting local energy conservation, this result implies that positive energy $+|E_o|$ moves from the field to external systems during the displacement operation $\hat{U}_s$ because the energy of the field around V_m is zero before the operation. Non-negativity of the total energy of the field ensures that $|E_o|$ is smaller than E_m, as discussed in [6]. The wave packets with negative energy begin to chase after the positive-energy wave packets which are generated by the first measurement. Figure 2 shows the energy extraction process (step (2)). The outside doughnut-shaped region represents the propagating wave packet generated by the measurement in step (1). The displacement operation $\hat{U}_s$ is performed in the measurement area V_m denoted by the circle in the figure. In figure 3, the average energy density is plotted immediately before the displacement operation at $t = T - 0$. We have zero energy around the area where $\hat{U}_s$ is performed. In figure 4, the average energy density after the operation is plotted. Negative energy density appears around V_m. In conclusion, a part of the escaped energy is effectively retrieved to the measurement area by this QET protocol.

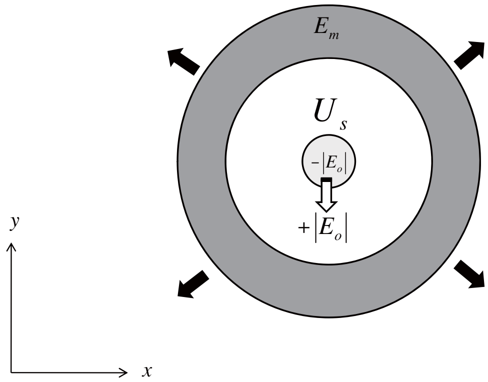

Figure 2. The second schematic diagram of QET in the xy-plane slice. The outside doughnut-shaped region represents the propagating wave packet generated by the measurement. By using the measurement result s, the displacement operation $\hat{U}_s$ is carried out in the measurement area V_m denoted by the circle. After the operation, a wave packet with negative energy $-|E_o|$ is generated, accompanying the extraction of positive energy $+|E_o|$ to external systems.

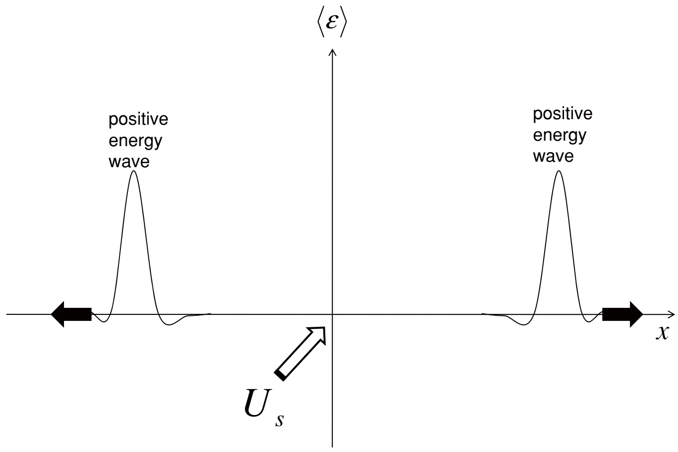

Figure 3. The average energy density is schematically plotted immediately before the displacement operation at $t = T - 0$. The positive-energy wave packet generated by the measurement has already escaped from the measurement area where $\hat{U}_s$ is performed.

In this discrete-variable case with the spin probe, the amount of extracted energy $|E_o|$ in equation (35) is suppressed by the exponential damping factor D_q in equation (36) when the magnitude of $\mathbf{a}_m$ and that of the infused energy E_m increase. This suppression becomes power damping for a continuous-variable teleportation with a probe harmonic oscillator, as seen in the next section.

4. Continuous-variable teleportation

In this section, we analyze a QET protocol with a continuous variable. The essential part of the protocol is the same as that discussed in section 3 except the probe system and the operation

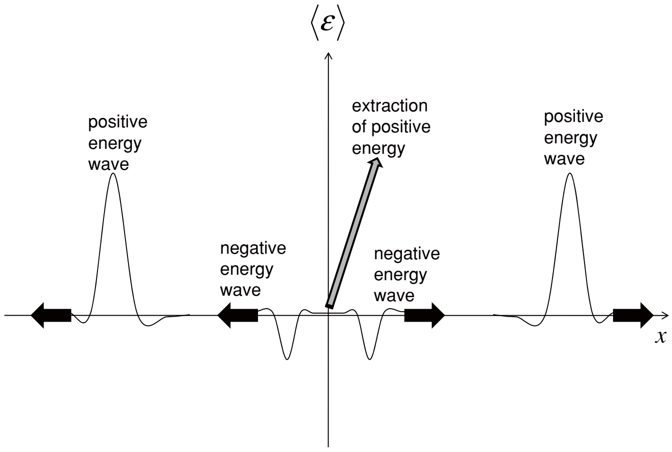

Figure 4. The average energy density is schematically plotted after the operation. The wave packet with negative energy $-|E_o|$ is generated by the extraction of positive energy $+|E_o|$ to external systems.

dependent on the measurement result. As the probe to measure the quantum fluctuation of the electric field, let us consider a harmonic oscillator. The free Hamiltonian is given by

$$\hat{H}_{ho} = \frac{1}{2}\hat{p}^2 + 2\hat{q}^2, \tag{37}$$

where $\hat{q}$ is the position operator and $\hat{p}$, the momentum operator. This oscillator couples with the electric field by the measurement interaction such that

$$\hat{H}'_m(t) = g(t)\hat{p}\hat{G},$$

where $g(t)$ is the time-dependent factor in equation (9) and $\hat{G}$, the Hermitian operator in equation (8). The initial state of the oscillator is the ground state $|g\rangle$ of H_{ho}. Introducing an eigenstate $|q\rangle$ of $\hat{q}$ with eigenvalue q, the ground state is described by

$$\langle q|g\rangle = \left(\frac{2}{\pi}\right)^{1/4} \exp[-q^2].$$

After the measurement interaction is switched off, we measure the position $\hat{q}$. The measurement operator for the measurement result q is computed as

$$\hat{M}_q = \langle q|\exp[-\mathrm{i}\hat{p}\hat{G}]|g\rangle.$$

More explicitly, $\hat{M}_q$ is written as

$$\hat{M}_q = \left(\frac{2}{\pi}\right)^{1/4} \exp[-(q-\hat{G})^2]. \tag{38}$$

The operator $\hat{M}_q$ satisfies the following relations:

$$\int_{-\infty}^{\infty} \hat{M}_q^\dagger \hat{M}_q \mathrm{d}q = 1, \tag{39}$$

$$\int_{-\infty}^{\infty} q\hat{M}_q^\dagger \hat{M}_q \mathrm{d}q = \hat{G}, \tag{40}$$

$$\int_{-\infty}^{\infty} q^2\hat{M}_q^\dagger \hat{M}_q \mathrm{d}q = \hat{G}^2 + \frac{1}{4}. \tag{41}$$

The state after the measurement is given by

$$\hat{\rho}'_m = \int \hat{M}_q |0\rangle\langle 0| \hat{M}_q^\dagger \mathrm{d}q.$$

In this measurement, energy is infused into the field as given by the protocol described in section 3. The expectation value of the infused energy density is evaluated as

$$\langle \hat{\varepsilon}(t,\mathbf{x})\rangle = \mathrm{Tr}[\hat{\rho}'_m \hat{\varepsilon}(t,\mathbf{x})] = \int \langle 0|\hat{M}_q^\dagger \hat{\varepsilon}(t,\mathbf{x})\hat{M}_q|0\rangle \mathrm{d}q, \tag{42}$$

where $\hat{\varepsilon}(t,\mathbf{x}) = \hat{U}^\dagger(t)\hat{\varepsilon}(\mathbf{x})\hat{U}(t)$. The value $\langle \hat{\varepsilon}(t,\mathbf{x})\rangle$ can be calculated by using the following integral formula:

$$\exp(-q^2) = \frac{1}{\sqrt{4\pi}} \int \mathrm{e}^{-\frac{p^2}{4} - \mathrm{i}pq} \mathrm{d}p. \tag{43}$$

From equation (43), the following relation is derived:

$$\begin{aligned} \hat{M}_q|0\rangle &= \left(\frac{2}{\pi}\right)^{1/4} \exp[-(q-\hat{G})^2]|0\rangle \\ &= \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{4\pi}} \int \mathrm{d}p \mathrm{e}^{-\frac{p^2}{4} - \mathrm{i}pq} \exp[\mathrm{i}p\hat{G}]|0\rangle. \end{aligned}$$

From equation (4), $\hat{M}_q|0\rangle$ is rewritten as

$$\hat{M}_q|0\rangle = \left(\frac{2}{\pi}\right)^{1/4} \frac{1}{\sqrt{4\pi}} \int \mathrm{d}p \mathrm{e}^{-\frac{p^2}{4} - \mathrm{i}p\left(q-\frac{\pi}{4}\right)} |(\mathbf{0}, p\mathbf{a}_m)\rangle. \tag{44}$$

Substituting equation (44) into equation (42) yields the following manipulation:

$$\begin{aligned} \langle \hat{\varepsilon}(t,\mathbf{x})\rangle &= \frac{1}{\sqrt{2\pi}} \int\int \mathrm{e}^{-\frac{\bar{p}^2}{4} - \frac{p^2}{4}} \langle (\mathbf{0}, \bar{p}\mathbf{a}_m)|\hat{\varepsilon}(t,\mathbf{x})|(\mathbf{0}, p\mathbf{a}_m)\rangle \left(\int \mathrm{e}^{\mathrm{i}(\bar{p}-p)\left(q-\frac{\pi}{4}\right)} \frac{\mathrm{d}q}{2\pi}\right) \mathrm{d}\bar{p}\mathrm{d}p \\ &= \frac{1}{\sqrt{2\pi}} \int \mathrm{d}p \mathrm{e}^{-\frac{p^2}{2}} \langle (\mathbf{0}, p\mathbf{a}_m)|\hat{\varepsilon}(t,\mathbf{x})|(\mathbf{0}, p\mathbf{a}_m)\rangle \\ &= \frac{1}{\sqrt{2\pi}} \int p^2 \mathrm{e}^{-\frac{p^2}{2}} \mathrm{d}p \langle (\mathbf{0}, \mathbf{a}_m)|\hat{\varepsilon}(t,\mathbf{x})|(\mathbf{0}, \mathbf{a}_m)\rangle \\ &= \langle (\mathbf{0}, \mathbf{a}_m)|\hat{\varepsilon}(t,\mathbf{x})|(\mathbf{0}, \mathbf{a}_m)\rangle, \end{aligned} \tag{45}$$

where we have used $\langle (\mathbf{0}, p\mathbf{a}_m)|\hat{\varepsilon}(t,\mathbf{x})|(\mathbf{0}, p\mathbf{a}_m)\rangle = p^2\langle (\mathbf{0}, \mathbf{a}_m)|\hat{\varepsilon}(t,\mathbf{x})|(\mathbf{0}, \mathbf{a}_m)\rangle$. From equation (45), we can show that the input energy $\int_{-\infty}^{\infty} \langle \hat{\varepsilon}(0,\mathbf{x})\rangle \, \mathrm{d}^3x$ in the measurement process is the same as that of the discrete-variable case described by equation (13) of section 3. Besides, in the same manner as the derivation of equation (17), it is possible from equation (45) to show the equation

$$\langle \hat{\varepsilon}(t,\mathbf{x})\rangle = \tfrac{1}{2}[\mathbf{\Pi}(t,\mathbf{x})^2 + \mathbf{b}(t,\mathbf{x})^2],$$

where $\mathbf{b}(t,\mathbf{x})$ and $\mathbf{\Pi}(t,\mathbf{x})$ are given by equation (18) and equation (19), respectively.

After the wave packets generated by the measurement escape from the measurement area, we execute at $t = T$, when the field has zero energy inside V_m, a local unitary operation dependent on the measurement result q given by

$$\hat{U}_q = \exp\left[\mathrm{i}q\theta' \int \mathbf{f}_o(\mathbf{x}) \cdot \hat{\mathbf{A}}(\mathbf{x}) \mathrm{d}^3x\right], \tag{46}$$

which is analogous to equation (20). The parameter θ' is real and fixed later. Then, the energy around the measurement area is evaluated as

$$E'_o = \int \mathrm{d}q \langle 0|\hat{M}_q^\dagger \hat{U}^\dagger(T)\hat{U}_q^\dagger \hat{H}_o \hat{U}_q \hat{U}(T)\hat{M}_q|0\rangle.$$

By using equations (39)–(41) and

$$\hat{U}^\dagger(T)\hat{U}_q^\dagger\hat{H}_o\hat{U}_q\hat{U}(T) = \int \mathbf{w}(\mathbf{x})\hat{\varepsilon}(t,\mathbf{x})\mathrm{d}^3x$$
$$+\theta' q\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{E}}(T,\mathbf{x})\mathrm{d}^3x + \frac{1}{2}\theta'^2q^2\int \mathbf{f}_o(\mathbf{x})^2\mathrm{d}^3x,$$

the average energy E_o' is rewritten, in a manner similar to that in section 3, as

$$E_o' = \theta'\eta' + \frac{\xi}{2}\theta'^2\langle 0|\left(\hat{G}^2+\frac{1}{4}\right)|0\rangle,$$

where ξ is given by equation (29) and η' is a real constant that is evaluated as

$$\begin{aligned}\eta' &= \langle 0|\left(\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{E}}(T,\mathbf{x})\mathrm{d}^3x\right)\hat{G}|0\rangle \\ &= -\langle 0|\left(\int \mathbf{f}_o(\mathbf{x})\cdot\hat{\mathbf{E}}(T,\mathbf{x})\mathrm{d}^3x\right)\left(\int \mathbf{a}_m(\mathbf{y})\cdot\hat{\mathbf{E}}(\mathbf{y})\mathrm{d}^3y\right)|0\rangle \\ &= \frac{1}{2}\int\int \partial_T^2\Delta(T,\mathbf{x}-\mathbf{y})\mathbf{f}_o(\mathbf{x})\cdot\mathbf{a}_m(\mathbf{y})\mathrm{d}^3x\mathrm{d}^3y. \end{aligned} \tag{47}$$

By fixing the parameter θ' as

$$\theta' = -\frac{\eta'}{\xi\langle 0|\left(\hat{G}^2+\frac{1}{4}\right)|0\rangle}, \tag{48}$$

the minimization of E_o' with respect to θ' is attained as

$$E_o' = -\frac{\eta'^2}{2\xi\langle 0|\left(\hat{G}^2+\frac{1}{4}\right)|0\rangle}.$$

By substituting equations (29) and (47) into the above equation, we obtain the final expression of E_o' as

$$E_o' = -D_{ho}\frac{\left[\int\int \partial_T^2\Delta(T,\mathbf{x}-\mathbf{y})\mathbf{f}_o(\mathbf{x})\cdot\mathbf{a}_m(\mathbf{y})\mathrm{d}^3x\mathrm{d}^3y\right]^2}{2\int \mathbf{f}_o(\mathbf{x})^2\mathrm{d}^3x}, \tag{49}$$

where D_{ho} is given by

$$D_{ho} = [\langle 0|(4\hat{G}^2+1)|0\rangle]^{-1} = \left[1+\frac{\pi^2}{4}+2\int |\mathbf{k}||\tilde{\mathbf{a}}_m(\mathbf{k})|^2\frac{\mathrm{d}^3k}{(2\pi)^3}\right]^{-1}. \tag{50}$$

It is noted that the form of E_o' in equation (49) is the same as that of E_o in equation (35), except for the form of the damping factor D_q. Unlike the exponential damping of D_q, the suppression factor D_{ho} is a power damping factor with respect to $|\tilde{\mathbf{a}}_m(\mathbf{k})|$. Because of this weak damping, the extracted energy $|E_o'|$ by this protocol does not vanish even for a large amplitude limit with $|\tilde{\mathbf{a}}_m(\mathbf{k})|\to\infty$ as

$$|E_o'| \sim \frac{\left|\int |\mathbf{k}_1|\cos(|\mathbf{k}_1|T)\,\mathbf{n}_m(\mathbf{k}_1)^*\cdot\tilde{\mathbf{f}}_o(\mathbf{k}_1)\frac{\mathrm{d}^3k_1}{(2\pi)^3}\right|^2}{4\left(\int\left|\tilde{\mathbf{f}}_o(\mathbf{k}_2)\right|^2\frac{\mathrm{d}^3k_2}{(2\pi)^3}\right)},$$

where $\tilde{\mathbf{f}}_o(\mathbf{k}) = \int \mathbf{f}_o(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}^3x$ and $\mathbf{n}_m(\mathbf{k})$ is a rescaled amplitude given by

$$\mathbf{n}_m(\mathbf{k}) = \frac{\tilde{\mathbf{a}}_m(\mathbf{k})}{\sqrt{\int |\mathbf{k}'|\,|\tilde{\mathbf{a}}_m(\mathbf{k}')|^2\frac{\mathrm{d}^3k'}{(2\pi)^3}}}.$$

As a conclusion, we can say that obtaining more information of the fluctuation leads to teleporting more energy. It is an important question what measurements and operations attain the maximum transporation rate of energy in the QET mechanism, however, remains unsolved yet.

5. Summary and discussion

We have analyzed in detail two protocols of QET for the electromagnetic field and shown that a part of the lost energy in the measurement can be retrieved by the use of the measurement result. The amount of energy infused by the measurement is the same in both the cases and it is given by equation (13). For the discrete-variable case, the measurement with the operator $\hat{M}_s$ given by equations (11) and (12) is performed. The amount of retrieval energy $|E_o|$ is given by equation (35) for the displacement operator in equation (20) with θ fixed in equation (33). For the continuous-variable case, the measurement with the operator $\hat{M}_s$ given by equation (38) is performed. The amount of energy retrieved $|E'_o|$ is given by equation (49) for the displacement operator in equation (46) with θ fixed in equation (48). For the large amplitude $|\tilde{\mathbf{a}}_m(\mathbf{k})|$ and the large energy input E_m, the continuous-variable teleportation is found to be more preferable than the discrete-variable teleportation. In the discrete-variable case, the amount of extracted energy is suppressed by the exponential damping factor D_q in equation (36) when the energy infused by the measurement increases. For the continuous-variable case, the suppression factor becomes the power damping factor D_{ho} in equation (50). Therefore, it is concluded that obtaining more information about the quantum fluctuation leads to teleporting more energy.

In future QET experiments, the separation between V_m and the region of the escaping wave packet with positive energy should not be so large to extract an observable amount of energy. To see this, let us take a large-separation limit as $T \gg |\mathbf{x} - \mathbf{y}|$ for $\mathbf{x}, \mathbf{y} \in \mathbf{V}_m$. Then, the amount of teleported energy decays as

$$|E_o| \propto \frac{1}{T^{12}},$$

in both the protocols. This rapid decay becomes one of the serious obstacles to observing the extraction of energy by QET. Therefore, for the best implementation of the QET in 1 + 3 dimensions, T should be of the same order as the measurement area size. From this viewpoint, QET protocols are more attractive for physical systems described effectively by (1 + 1)-dimensional massless field models that have slower-decay properties ($E_o \propto T^{-4}$) of the transported energy for a large separation T, as discussed in [6, 7].

Acknowledgments

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